Casimir effect with nonlocal boundary interactions

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Abstract

We derive a general expression for the Casimir energy corresponding to two flat parallel mirrors in $d+1$ dimensions, described by nonlocal interaction potentials. For a real scalar field, the interaction with the mirrors is implemented by a term which is a quadratic form in the field, with a nonlocal kernel. The resulting expression for the energy is a function of the parameters that define the nonlocal kernel. We show that the general expression has the correct limit in the zero width case, and also present the exact solution for a particular case.

The increasing interest in the Casimir effect [1] is a natural consequence of the availability of new precision experiments, which pose an important pressure to continuously refine and improve the existing calculations. This evolution manifests itself at different levels; one of them amounts to coping with situations where the geometry of the mirrors is more complex, albeit with an idealized description of their material properties, i.e., they are regarded as mathematical surfaces occupied by perfect conductors. Another level, which has recently received much attention, is the use of a more accurate description of the mirrors, including corrections that represent their departure from exactly conducting surfaces: rugosity, finite temperature and conductivity, as well as finite width. The latter is usually dealt with by the introduction of a 'space dependent mass term' whereby the scalar field becomes very massive at the locii of the mirrors [2]. For the case of a real scalar field φ in $d+1$ spacetime dimensions, and a single flat mirror centered at $x_d = 0$, the local

Euclidean action, S_{local}, for this kind of term is:

$$S_{local}(\varphi) \;=\; \frac{1}{2} \int d^d x_\parallel \int dx_d \, V_\epsilon(x_d) [\varphi(x_\parallel, x_d)]^2 \;, \tag{1}$$

where $x_\parallel$ denotes the time (x_0) as well as the $d-1$ spatial coordinates parallel to the mirror (which we shall denote by $\mathbf{x}_\parallel$). The local potential $V_\epsilon(x_d)$ is a positive function concentrated around 0, with a width of size ϵ. The perfect mirror case is approached when that size tends to zero and its strength becomes infinite; namely, $V_\epsilon(x_d) \to g\,\delta(x_d)$. $g \to \infty$. This limit is a delicate step, since it usually introduces divergences [3] that may be difficult to deal with in a setup that used idealized boundary conditions only.

Local interaction terms have been extended to include a non trivial dependence on the parallel coordinates [4, 5]. Translation invariance along them, necessarily implies a spatial nonlocality:

$$S'_{local}(\varphi) \;=\; \frac{1}{2} \int d^d x_\parallel \int d^d x'_\parallel \int dx_d \; \varphi(x_\parallel, x_d) V_\epsilon(x_\parallel - x'_\parallel; x_d) \varphi(x'_\parallel, x_d) \;, \tag{2}$$

but a Fourier transformation in $x_\parallel$, $x'_\parallel$ yields a local expression in the mixed momentum ($k_\parallel$) and coordinate (x_d) representation:

$$S'_{local}(\varphi) \;=\; \frac{1}{2} \int \frac{d^d k_\parallel}{(2\pi)^d} \int dx_d \; \widetilde{\varphi}^*(k_\parallel, x_d) \, \widetilde{V}_\epsilon(k_\parallel; x_d) \, \widetilde{\varphi}(k_\parallel, x_d) \;, \tag{3}$$

where the tildes denote the Fourier transformed of the corresponding object.

Note that the resulting interaction term, $S'_{local}(\varphi)$, is still assumed to be local in x_d, and $\widetilde{V}_\epsilon(k_\parallel; x_d)$ is concentrated around $x_d = 0$, on a region of size $\sim \epsilon$. However, except for the case of a zero-width mirror, a potential which is local in x_d can only be an approximate description of the interaction with a real material. Indeed, as explained in [6], one should in general use interactions that also include 'spatial dispersion' in x_d, i.e., nonlocality in the normal coordinates. As also shown in [6], in spite of the nonlocality of the interaction, one may nevertheless use a Lifshitz formula [9] for the Casimir energy, since that formula depends on the reflection coefficients at the media surfaces, and they may be defined also for nonlocal media, under quite general assumptions. Interesting results have also been obtained using the boundary state formalism [7].

We note in pass that the relevance of nonlocal media to Casimir-like effects has also been appreciated in other contexts, like String Theory [8].

In spite of the validity and usefulness of Lifshitz formula for nonlocal media, we believe that it would be important to have an alternative expression for the Casimir energy, where its dependence on the details defining the nonlocal media were more explicit.

One might also be interested in situations where the material is not exactly confined to the region between two surfaces, but rather is concentrated on a region, with a non zero (albeit rapidly vanishing) density outside of that region. This is a situation where the use of reflection coefficients, albeit still possible, becomes nevertheless problematic.

To confront those problems, we first define the setup: for a mirror centered at an arbitrary position $x_d = b$, we shall use an interaction term $S_I^{(b)}$ given by:

$$\begin{aligned} S_I^{(b)}(\varphi) &= \frac{1}{2}\int \frac{d^dk_\parallel}{(2\pi)^d}\int dx_d \int dx'_d \\ &\times \widetilde{\varphi}^*(k_\parallel, x_d)\, \widetilde{V}_\epsilon(k_\parallel; x_d - b, x'_d - b)\, \widetilde{\varphi}(k_\parallel, x'_d)\ . \end{aligned} \tag{4}$$

The kernel $\widetilde{V}_\epsilon(k_\parallel; x_d, x'_d)$ is not invariant under translations in the normal direction ($x_d \to x_d + h$, $x'_d \to x'_d + h$); besides, we still assume it to be concentrated around $x_d = 0$ and $x'_d = 0$. This concentration may be used to write a convenient expansion for the kernel, based on the introduction of $\left[\psi_n^{(\epsilon)}(x_d)\right]_n$, an orthonormal basis of functions of the normal coordinate, obeying the boundary conditions that follow from the microscopic model [1].

Then, without any loss of generality, the nonlocal kernel will be expanded as follows:

$$\widetilde{V}_\epsilon(k_\parallel; x_d, x'_d) = \sum_{m,n} C_{mn}(k_\parallel, \epsilon)\, \psi_m^{(\epsilon)}(x_d)\, \psi_n^{(\epsilon)*}(x'_d)\ , \tag{5}$$

where $C_{mn}(k_\parallel, \epsilon) = C^*_{nm}(k_\parallel, \epsilon)$, from the reality of the action.

Being a nonlocal term, one has to rephrase the properties it should have to behave as a (generalized) mass term, in the sense that it favours the vanishing of the field around the region where it is different from zero. It is clear then that the quadratic form (5) has to be definite positive (in the space generated by the basis); this amounts to a non-trivial relation for the C_{mn} matrix.

Let us illustrate the previous construction with two examples. Firstly, we consider a model where $S_I^{(b)}$ emerges from the linear coupling of φ to a microscopic real scalar field $\xi(x)$, which is confined to the region $|x_d - b| \leq \epsilon/2$ ($x_\parallel$: arbitrary), satisfying Dirichlet boundary conditions at $x_d = b \pm \epsilon/2$. It is sufficient to deal with $b = 0$, since the general case is obtained by a translation of the kernel. Following a generalization of the approach of [10], we see that,

[1] For example, they could satisfy Dirichlet boundary conditions at $x_d = \pm\epsilon/2$. But they are not necessarily of compact support; they could for example be exponentially decaying functions with a typical dispersion $\sim \epsilon$

in the functional formalism, $S_I^{(0)}$ may be written as follows:

$$e^{-S_I^{(0)}(\varphi)} \;=\; \frac{\int \mathcal{D}\xi \, e^{-S_m(\xi) + ig\int d^d x_\parallel \int_{-\epsilon/2}^{+\epsilon/2} dx_d \, \xi(x_\parallel, x_d)\, \varphi(x_\parallel, x_d)}}{\int \mathcal{D}\xi \, e^{-S_m(\xi)}} \;, \tag{6}$$

where g is a coupling constant and S_m is the action for the microscopic field. The matter field ξ may have a self-interaction, controlled by an independent coupling constant (implicit in S_m).

To proceed, we denote by $W(J)$ the generating functional of connected correlation functions of ξ, related to $\mathcal{Z}(J)$, the one for the full correlation functions:

$$\mathcal{Z}(J) \;=\; \int \mathcal{D}\xi \, e^{-S_m(\xi) + \int d^d x_\parallel \int_{-\epsilon/2}^{+\epsilon/2} dx_d \, J(x_\parallel, x_d)\xi(x_\parallel, x_d)} \;, \tag{7}$$

by $W = \ln \mathcal{Z}$. The current J is confined to the same region as ξ, but it has free boundary conditions. We then have that $S_I(\varphi) = -W[i\, g\, \varphi(x)]$. On the other hand, since only the quadratic part in φ will be retained [2],

$$\begin{aligned} S_I^{(0)}(\varphi) &= -W\big[i\, g\, \varphi(x)\big] \\ &\simeq \frac{1}{2} g^2 \int d^{d+1}x \int d^{d+1}x' \, \varphi(x) W^{(2)}(x, x') \varphi(x') \;, \end{aligned} \tag{8}$$

where $W^{(2)}$ is the connected 2-point function.

Now, assuming that S_m is translation invariant along $x_\parallel$, we immediately identify the nonlocal kernel:

$$\widetilde{V}_\epsilon(k_\parallel; x_d, x'_d) \;=\; g^2 \, \widetilde{W}^{(2)}(k_\parallel; x_d, x'_d) \;. \tag{9}$$

Besides, since ξ satisfies Dirichlet boundary conditions, we have:

$$\widetilde{W}^{(2)}(k_\parallel; \pm\epsilon/2, x'_d) \;=\; \widetilde{W}^{(2)}(k_\parallel; x_d, \pm\epsilon/2) \;=\; 0 \;. \tag{10}$$

Then we have the expansion:

$$\widetilde{V}_\epsilon^{(0)}(k_\parallel; x_d, x'_d) \;=\; \sum_{m,n} \psi_m^{(\epsilon)}(x_d) \, C_{mn}(k_\parallel, \epsilon) \, \psi_m^{(\epsilon)*}(x_d) \tag{11}$$

where the orthonormal functions are given by:

$$\psi_n^{(\epsilon)}(x_d) \;=\; \sqrt{\frac{2}{\epsilon}} \times \begin{cases} \sin(\frac{n\pi x_d}{\epsilon}) & \text{if} \quad n = 2k, \qquad (k = 1, 2, \ldots) \\ \cos(\frac{n\pi x_d}{\epsilon}) & \text{if} \quad n = 2k+1, \quad (k = 0, 1, \ldots) \;. \end{cases} \tag{12}$$

[2] The media are assumed to be linear.

The precise form of $C_{mn}(k_\parallel,\epsilon)$ depends on the action S_m. If it is a free action we have the *diagonal* expression:

$$C_{mn}(k_\parallel,\epsilon) \;=\; \frac{g^2\delta_{mn}}{(\frac{n\pi}{\epsilon})^2 + k_\parallel^2 + \mu^2}\;, \tag{13}$$

where μ is the mass of the microscopic field.

As an alternative example, we consider the case of a charged field ξ, $\bar{\xi}$, coupled quadratically to the real field φ. In this case, the analogue expression to (6) would be:

$$e^{-S_I^{(0)}(\varphi)} \;=\; \frac{\int \mathcal{D}\xi\mathcal{D}\bar{\xi}\, e^{-S_m(\bar{\xi},\xi) + g\int d^dx_\parallel \int_{-\epsilon/2}^{+\epsilon/2} dx_d\, \bar{\xi}(x_\parallel,x_d)\varphi(x_\parallel,x_d)\xi(x_\parallel,x_d)}}{\int \mathcal{D}\xi\mathcal{D}\bar{\xi}\, e^{-S_m(\bar{\xi},\xi)}}\;, \tag{14}$$

where, to simplify the treatment, we assume S_m to be quadratic:

$$S_m(\bar{\xi},\xi) \;=\; \int d^dx_\parallel \int_{-\epsilon/2}^{+\epsilon/2} dx_d \big[\partial\bar{\xi}\partial\xi \,+\, \mu^2\bar{\xi}\xi\big]\;. \tag{15}$$

The integral can be formally performed, but the result is a functional determinant. Since we use a quadratic approximation for $S_I^{(0)}$, we only need it up to that order in φ. The corresponding contribution is just a 1-loop diagram with two legs. Translation invariance along the parallel coordinates suggest the use of a mixed Fourier representation:

$$\tilde{V}_\epsilon(k_\parallel;x_d,x_d') \;=\; g^2 \int \frac{d^dp_\parallel}{(2\pi)^d}\,\tilde{G}(p_\parallel;x_d,x_d')\tilde{G}(p_\parallel+k_\parallel;x_d',x_d) \tag{16}$$

where $\tilde{G}(p_\parallel;x_d,x_d')$ is the microscopic field propagator in the mixed representation. Rather than evaluating the actual form of the nonlocal term for this model, we just want to show that the boundary conditions for the nonlocal kernel are determined from the ones we impose on the microscopic field: assuming, for example, that this field satisfies Dirichlet boundary conditions at the boundaries of the mirror, from (16) we derive for $\tilde{V}_\epsilon$ the same kind of condition. Namely, the kernel vanishes when $x_d = \pm\epsilon/2$ or $x_d' = \pm\epsilon/2$. Thus, also in this case the model produces a nonlocal kernel with the structure of (11) (the same basis), with different coefficients C_{mn}.

Before evaluating the energy, let us briefly examine the boundary conditions that follow from the nonlocal term, in a concrete case. To that end, we consider the real-time version of the equations of motion for a free massless scalar field coupled to a nonlocal potential centered at $x_d = 0$. Assuming,

for the sake of simplicity, $d = 1$, and $C_{mn} = C_{mn}(\epsilon)$ (independent of ω), the equation of motion becomes:

$$\Box\varphi(x_0, x_1) \;=\; -\int dx_1'\, V_\epsilon(x_1, x_1')\, \varphi(x_0, x_1') \;. \tag{17}$$

Fourier transforming in time,

$$(-\partial_1^2 - \omega^2)\tilde{\varphi}(\omega, x_1') \;=\; -\int dx_1'\, V_\epsilon(x_1, x_1')\, \tilde{\varphi}(\omega, x_1') \;. \tag{18}$$

Then we multiply both sides of the equation above by $\psi_m^{(\epsilon)*}(x_1)$ and integrate over x_1, to obtain:

$$\langle\psi_m^{(\epsilon)}|(-\partial_1^2 - \omega^2)|\tilde{\varphi}(\omega)\rangle \;=\; -\sum_n C_{mn}(\epsilon)\, \langle\psi_n^{(\epsilon)}|\tilde{\varphi}(\omega)\rangle \;, \tag{19}$$

where Dirac's bracket notation denotes the scalar product of functions of x_d. Let us assume, for the sake of simplicity, that the proper basis is (12). Then $\langle\psi_m^{(\epsilon)}|(-\partial_1^2)|\tilde{\varphi}\rangle \;=\; (\frac{m\pi}{\epsilon})^2\, \langle\psi_m^{(\epsilon)}|\tilde{\varphi}\rangle$ and, as a consequence:

$$\sum_n C_{mn}(\epsilon)\, \alpha_n \;=\; \left[\omega^2 - (\frac{m\pi}{\epsilon})^2\right] \alpha_m, \tag{20}$$

where $\alpha_n \equiv \langle\psi_n^{(\epsilon)}|\tilde{\varphi}(k_\parallel)\rangle$. Thus,

$$\sum_n C_{mn}(\epsilon)\, \alpha_m^*\, \alpha_n \;=\; \sum_m \left[\omega^2 - (\frac{m\pi}{\epsilon})^2\right] |\alpha_m|^2 \;. \tag{21}$$

This means that, to have a solution, the α_n coefficients diagonalize C, and, since the quadratic form on the left hand side is positive, we α_m vanishes whenever $\omega^2 < (\frac{m\pi}{\epsilon})^2$.

This means, in particular, that for $\omega^2 < (\frac{\pi}{\epsilon})^2$, all the coefficients vanish: the field vanishes when $|x_1| < \frac{\epsilon}{2}$. Of course, things are different if, for example: $(\frac{\pi}{\epsilon})^2 < \omega^2 < (\frac{2\pi}{\epsilon})^2$, then only the first coefficient may be different from 0.

An interesting case is that of a C_{mn} which is a finite matrix: one that vanishes for $m > N$ of , say. An extreme case is $N = 1$: there are then only two regimes, depending on whether ω^2 is bigger or smaller than $(\frac{\pi}{a})^2$. In the former case, φ in orthogonal to $\psi_1^{(\epsilon)}$. This implies that *has at least one node* in the $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ interval. This is the manifestation of a Dirichlet-like boundary condition in this context, which of course will only hold true up to certain values of ω^2. For bigger values, the previous condition is relaxed and the mirror is transparent.

Let us now evaluate the Casimir energy, discarding terms that are independent of the distance between mirrors (and do not contribute to the force). For two mirrors, one at $x_d = 0$ and the other at $x_d = a$, the total action S is $S(\varphi) = S_0(\varphi) + S_I(\varphi)$ where $S_0 \equiv \frac{1}{2}\int d^{d+1}x\, \partial_\mu\varphi\partial_\mu\varphi$, and $S_I(\varphi) = S_I^{(0)} + S_I^{(a)}$.

Since S is a quadratic in the fields, it is immediate to find an expression for the vacuum energy E_0, in terms of the determinant of the corresponding kernel defining the quadratic form. Since we have translation invariance along $\mathbf{x}_\parallel$, we use the energy per unit area, $\mathcal{E}_0$, and take advantage of the Fourier transformation to obtain:

$$\mathcal{E}_0 \;=\; \frac{1}{2}\int \frac{d^dk_\parallel}{(2\pi)^d}\, \mathrm{Tr}\ln\widetilde{\mathcal{K}}\;, \tag{22}$$

where $\widetilde{\mathcal{K}}$ is an operator acting on functions of x_d, whose matrix elements are:

$$\begin{aligned}\widetilde{\mathcal{K}}(x_d,x'_d) \;&=\; \widetilde{\mathcal{K}}_0(x_d,x'_d) \,+\, \widetilde{V}_\epsilon^{(0)}(k_\parallel;x_d,{x'}_d) \\ &+\; \widetilde{V}_\epsilon^{(a)}(k_\parallel;x_d,{x'}_d)\;.\end{aligned} \tag{23}$$

where $\mathcal{K}_0(x_d,x'_d) \equiv (-\partial_d^2 + k_\parallel^2)\delta(x_d - x'_d)$. The trace operation, denoted by 'Tr' refers to the trace in the space of functions depending on x_d.

The expression above contains three contributions which, to calculate the Casimir force between the two mirrors, are irrelevant. One of them, $\mathcal{E}_0^{vac}$, corresponds to the vacuum energy density in the absence of mirrors

$$\mathcal{E}_0^{vac} \;=\; \frac{1}{2}\int \frac{d^dk_\parallel}{(2\pi)^d}\, \mathrm{Tr}\ln\widetilde{\mathcal{K}}_0\;. \tag{24}$$

The other two, denoted by $\mathcal{E}_0^{(0)}$ and $\mathcal{E}_0^{(a)}$, are the mirrors' self-energies (and therefore we shall discard them). They have the form:

$$\mathcal{E}_0^{(b)} \;=\; \frac{1}{2}\int \frac{d^dk_\parallel}{(2\pi)^d}\, \mathrm{Tr}\ln\left(I + W^{(b)}\right)\;. \tag{25}$$

where $b = 0$, a, and $W^{(b)}$ denotes an operator, acting on the same space as above, and defined by:

$$W^{(b)} \;=\; \widetilde{\mathcal{K}}_0^{-1}\, V^{(b)}\;. \tag{26}$$

$V^{(0)}$, $V^{(a)}$ have non trivial matrix elements in smaller spaces, namely, the ones generated by the basis functions sitting at each mirror. This fact can be used to show that the trace operation above may be taken, for each term, using only the respective basis at $x_d = 0$ and $x_d = a$ (the trace operation over the complement vanishes).

Using matrix elements defined with the functions $\psi_n^{(\epsilon)}$ and $\phi_n^{(\epsilon)}$, such that $\phi_n^{(\epsilon)}(x_d) \equiv \psi_n^{(\epsilon)}(x_d - a)$:

$$\begin{aligned} W^{(0)}_{mn} &= \langle \psi_m^{(\epsilon)} | \widetilde{\mathcal{K}}_0^{-1} \, V^{(0)} | \psi_n^{(\epsilon)} \rangle \\ W^{(a)}_{mn} &= \langle \phi_m^{(\epsilon)} | \widetilde{\mathcal{K}}_0^{-1} \, V^{(a)} | \phi_n^{(\epsilon)} \rangle \; . \end{aligned} \tag{27}$$

I denotes the identity operator, so that: $I_{mn} = \delta_{mn}$. A simple shift of variables leads to $\mathcal{E}_0^{(0)} = \mathcal{E}_0^{(a)}$, as it should be.

After extracting the previous three contributions, we obtain a subtracted energy density, $\tilde{\mathcal{E}}_0$, which by some straightforward algebra may be put in the form:

$$\tilde{\mathcal{E}}_0 \;=\; \frac{1}{2} \int \frac{d^d k_\parallel}{(2\pi)^d} \, \mathrm{Tr} \ln \big(I - \mathcal{O} \big) \; , \tag{28}$$

where $\mathcal{O}$ is an operator whose matrix elements may be given in terms of the $\psi_n^{(\epsilon)}$ basis, as follows:

$$\mathcal{O}_{mn} \;=\; \sum_{p,q,r} U^{(0)}_{mp} C_{pq} U^{(a)}_{qr} C_{rn} \; , \tag{29}$$

where

$$\begin{aligned} U_{mn} &= \langle \psi_m^{(\epsilon)} | \big[\widetilde{\mathcal{K}}_0 \, + \, V_\epsilon^{(0)} \big]^{-1} | \phi_n^{(\epsilon)} \rangle \\ U^{(a)}_{mn} &= \langle \phi_m^{(\epsilon)} | \big[\widetilde{\mathcal{K}}_0 \, + \, V_\epsilon^{(a)} \big]^{-1} | \psi_n^{(\epsilon)} \rangle \;=\; U^{(0)}_{mn} \;\equiv\; U_{mn} \; . \end{aligned} \tag{30}$$

Taking into account the previous relations,

$$\tilde{\mathcal{E}}_0 \;=\; \frac{1}{2} \int \frac{d^d k_\parallel}{(2\pi)^d} \, \mathrm{Tr} \ln \big(I - U \, C \, U \, C \big) \; , \tag{31}$$

which is a nonlocal version, for flat, identical mirrors, of the equation derived in [11] (see also [12]).

Note that U_{mn} can be written more explicitly, by performing an expansion in powers of $V_\epsilon^{(0)}$. Defining: $\Delta_{mn} \equiv \langle \psi_m^{(\epsilon)} | \big[\mathcal{K}_0 \big]^{-1} | \psi_n^{(\epsilon)} \rangle$ and $\Gamma_{mn} \equiv \langle \psi_m^{(\epsilon)} | \big[\mathcal{K}_0 \big]^{-1} | \phi_n^{(\epsilon)} \rangle$, we see that:

$$\begin{aligned} U^{(0)}_{mn} &= \Gamma_{mn} - \Delta_{mp} C_{pq} \Gamma_{qn} \\ &+ \Delta_{mp} C_{pq} \Delta_{qr} C_{rs} \Gamma_{sn} + \ldots \end{aligned} \tag{32}$$

(we used Einstein's summation convention). Then we see that:

$$U \;=\; \big(I + \Delta C \big)^{-1} \Gamma \; . \tag{33}$$

Let us check that the expressions above do yield the proper answer when the limit corresponding to the case of perfect mirrors is taken. This is, in the present formalism, tantamount to:

$$\tilde{V}_\epsilon^{(0)}(k_\parallel; x_d, x'_d) \;=\; \lambda\, \delta(x_d)\, \delta(x'_d) \;, \tag{34}$$

(sharp boundary conditions) and then $\lambda \to \infty$ (strong boundary conditions). One immediately gets, from (34), the matrix elements:

$$C_{mn} \;=\; \lambda\, \psi_m^{(\epsilon)}(0)\, \psi_m^{(\epsilon)*}(0) \;. \tag{35}$$

Inserting this into (29), we obtain:

$$\tilde{\mathcal{E}}_0 \;=\; \frac{1}{2} \int \frac{d^d k_\parallel}{(2\pi)^d} \ln \Big[1 \;-\; \big(\frac{\lambda}{2k_\parallel \,+\, \lambda} \big)^2 e^{-2k_\parallel a} \Big] \;, \tag{36}$$

which in the strong limit, and for $d = 3$ yields:

$$\begin{aligned} \tilde{\mathcal{E}}_0 \;&=\; \frac{1}{2} \int \frac{d^3 k_\parallel}{(2\pi)^3} \ln \Big(1 \;-\; e^{-2k_\parallel a} \Big) \\ &=\; -\frac{\pi^2}{1440\, a^3} \;, \end{aligned} \tag{37}$$

which is the proper result.

Finally we consider a truly nonlocal example:

$$C_{mn}(k_\parallel, \epsilon) \;=\; \delta_{m0} \delta_{n0}\, \lambda(k_\parallel) \;, \tag{38}$$

where $\lambda(k_\parallel) > 0$. It corresponds to a 1×1 matrix C in the space generated by the basis functions. In this case, we find that:

$$\mathcal{O}_{mn} \;=\; \big[\lambda(k_\parallel) \big]^2\, U_{00} U_{m0} \delta_{n0} \;. \tag{39}$$

For this particular case, the $\mathcal{O}$ matrix has rank 1, so only one of its eigenvalues is different from 0, and the trace of the log may be evaluated exactly. The Casimir energy becomes:

$$\tilde{\mathcal{E}}_0 \;=\; \frac{1}{2} \int \frac{d^d k_\parallel}{(2\pi)^d} \, \ln \Big[1 - \lambda^2(k_\parallel) \big(U_{00} \big)^2 \Big] \;. \tag{40}$$

We may produce a more explicit expression for the matrix elements of U:

$$U_{00}^{(0)} = \big[1 + \lambda(k_\parallel)\, \Delta_{00} \big]^{-1} \Gamma_{00} \;, \tag{41}$$

where:

$$\Delta_{00} \;=\; \int \frac{dk_d}{2\pi}\,\frac{\left|\langle\psi_0^{(\epsilon)}|k_d\rangle\right|^2}{k_d^2+k_\parallel^2} \tag{42}$$

($|k_d\rangle$ is the plane wave ket).

The remaining object, Γ_{00} is:

$$\Gamma_{00} \;=\; \int \frac{dk_d}{2\pi}\,e^{ik_d a}\,\frac{\left|\langle\psi_0^{(\epsilon)}|k_d\rangle\right|^2}{k_d^2+k_\parallel^2}\;. \tag{43}$$

Equipped with the previous expressions, we may calculate the Casimir energy for a particular basis element. An interesting example is the choice of the exponentially localized function

$$\psi_0^{(\epsilon)} = \frac{e^{-\frac{x^2}{2\,\epsilon^2}}}{\pi^{1/4}\,\epsilon^{1/2}}\;, \tag{44}$$

since it allows one to evaluate the integrals above exactly. The result for $\tilde{\mathcal{E}}_0$, in d spatial dimensions, may be written in terms of an integral involving a function G

$$\tilde{\mathcal{E}}_0(a,\epsilon) \;=\; \frac{1}{\Gamma\left(\frac{d}{2}\right)2^d\pi^{d/2}a^d}\int_0^\infty dp\,p^{d-1}\,\ln\left\{1-\Big[G\big(p;\frac{2\epsilon}{a}\,,\,a\lambda(\frac{p}{a})\big)\Big]^2\right\}\;, \tag{45}$$

depending on dimensionless parameters. It is given explicitly by:

$$G(p;x,l) \;=\; \frac{e^{-p}\,\mathrm{erfc}(x\,p-\frac{1}{2x})+\,e^{p}\,\mathrm{erfc}(x\,p+\frac{1}{2x})}{\frac{p\,e^{-x^2p^2}}{2\sqrt{\pi}\,l}+2\,\mathrm{erfc}\,(xp)}\;, \tag{46}$$

where erfc is the complementary error function.

It is immediate to check that the Casimir energy for the perfect mirror case is reproduced when $\frac{\epsilon}{a}\to 0$ and $\lambda\to\infty$. On the other hand, when that limit is taken for a *finite* λ, the result has the same form as in the local case, but with a (finite) renormalization for λ. Indeed, if λ_{local} denotes the coupling constant in the local δ-potential case, the energies agree for $\lambda=\frac{\lambda_{local}}{8\sqrt{\pi}}$.

For finite values of $\frac{\epsilon}{a}$, we have the interesting phenomenon that the corrections are not analytical in that variable. Indeed, one can see that the corrections to the zero-width case are proportional to a factor $e^{-(\frac{a}{2\epsilon})^2}$.

To make the comparison with the local case more explicit we present, in Figure 1, the plots of the Casimir energies corresponding to the local and nonlocal cases, for $d=1$. The local potential is chosen so that it agrees with the nonlocal one when $\epsilon\to 0$ ($\lambda_{local}=8\sqrt{\pi}\lambda$):

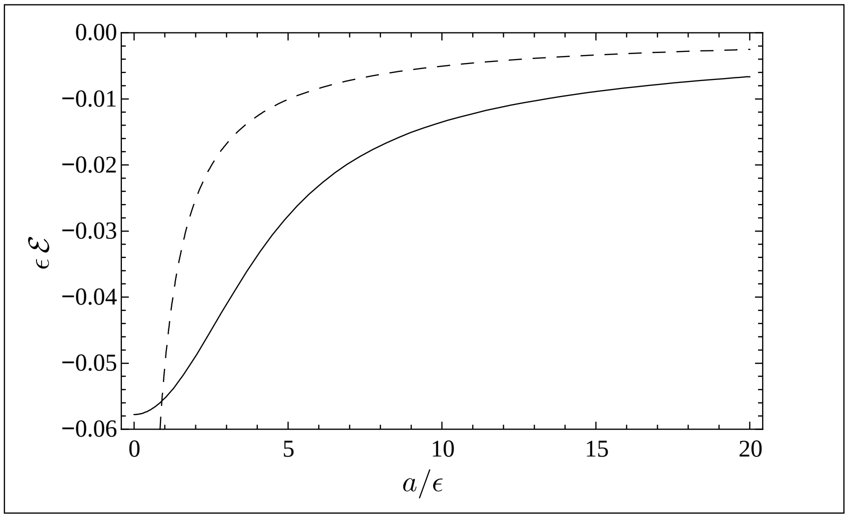

Figure 1: Casimir energies corresponding to the nonlocal (continuous line) and local (dashed line) cases, as a function of $b \equiv \frac{a}{\epsilon}$ for $\epsilon\lambda \equiv \frac{1}{8\sqrt{\pi}}$.

A remarkable fact, that can be observed in the plot, is that the Casimir energy for the nonlocal case becomes *finite* when the distance between the mirrors tends to zero. This is a manifestation of the fact that the nonlocality softens the UV behaviour of the system. Yet another consequence of the same effect is that the integral over $k_\parallel$ for the energy of a single mirror, $\mathcal{E}_0^{(b)}$ (see Eq. (25)), has a better UV behaviour that its local counterpart. In particular, for $d = 1$, a simple calculation shows that it becomes:

$$\mathcal{E}_0^{(b)} \;=\; \frac{1}{2\pi}\int_0^\infty dk\;\ln\left[1 + \frac{4\lambda\sqrt{\pi}}{k}\;\; e^{4\,\epsilon^2 k^2}\,\mathrm{erfc}\,(2\epsilon\, k)\right]\;, \tag{47}$$

which is convergent for any $\epsilon > 0$ (we recall that its local counterpart is logarithmically divergent [3]).

We conclude by noting that, as shown in the examples above, nonlocal potentials can be used to impose boundary conditions in finite size mirrors, and they becomes automatically frequency dependent. Also, in spite of their seemingly complex structure, a general expression for the energy may be derived, which contains some new interesting features: the non-analytic behaviour of its small-width expansion and a softer UV behaviour.

In spite of the above, the perfect mirror limit is still properly reproduced. Besides, when the distance between mirrors is of the order of ϵ, the Casimir force vanishes, rather than becoming infinite, as it happens in the local case.

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